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EXTREME ORDER STATISTICS PLOT VERSUS QUANTILE QUANTILE PLOT: NONPARAMETRIC VISUALIZATION FOR A DATA

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ABSTRACT

New plots are proposed based on minimum and maximum order statistics that is visually appealing, easy to understand, stable at extreme tails and capture all information about the distribution of the data. The minimum and maximum plots give more weights to the data at the extreme tails unlike quantile quantile plot. Therefore, it can be considered these plots as a completeness of the quantile quantile plot. The minimum and maximum plots are used to obtain a nonparametric visualization for the Gumbel and Weibull distributions. Moreover, the minimum and maximum normal plots are introduced and compared with quantile quantile plot. The new plots have advantage to be applied to discrete distributions.

KEYWORDS: Extreme Values, Gumbel Distribution, Order Statistics, Q-Q Plot, Weibull Plot

Msc2010 Classification: 62 Statistics (62gxx)

1. INTRODUCTION

Graphical presentation of data is an important tool in sciences. Good graph reflects a great deal of information and can be used to extract new conclusions while bad graph can be misleading and confusing. Given a random sample of univariate data points, a pertinent question is whether this sample comes from some specified distribution F. Decision techniques are based on how close the empirical distribution of the sample and the distribution F are for some sample size n.

Quantile-quantile (Q-Q) plot is commonly used device to graphically and informally test the goodness-of-fit of a sample in an exploratory way. It is used to plot the sample quantiles against the theoretical quantiles or other sample quantiles and then a visual check is made to see whether or not the points are close to a straight line; see, Chambers et al (1983), Cleveland (1994), Scott (1992) and Cleveland and McGill (1988). The pattern of points in the plot is used to compare the shapes of distributions, providing a graphical view of how properties such as location, scale and skewness. The use of Q–Q plots to compare two samples of data can be viewed as a non parametric approach to comparing their underlying distributions. A Q–Q plot is generally a more powerful approach to do this than the common technique of comparing histogram of the two samples, but requires more skill to interpret; see, Makkonen (2008) and Wilk and Gnanadesikan (1968).

Extreme order statistics plots are proposed based on minimum and maximum order statistics from population of size k (Min-Max plots). The plots can be done in parametric and nonparametric ways. The Min-Max plots give more weights to the data at the extreme tails of the distribution. Therefore, these plots will complete the picture of the data with QQ plot especially at the extreme tails of the distribution. Min-Max plots are used to obtain nonparametric characterization for the Gumbel and Weibull distributions. Since a variety of estimation and inferential procedures in the practice depends on the assumption of normality, the Min-normal plot and Max-normal plot are introduced and compared with Q-Q plot. These plots characterize and capture all information about the whole distribution of the data. The pattern of the points in the Min-Max plots is used to compare the shapes of distributions non-parametrically. Min-Max plots are used to plot the data against theoretical extreme order statistics or sample extreme order statistics and then a visual check is made to see whether or not the points are close to a straight line but the Min-Max plots have more stability at the tails of the distribution than Q-Q plot.

The extreme order statistics plots and their characterization to probability distributions are derived in Section 2. The Min and Max normal plots are introduced in Section 3. The nonparametric visualization for Gumbel and Weibull distributions is proposed in Section 4. An extension of Min and Max plots to discrete distributions are introduced in section 5. Two applications are studied in Section 6. Section 7 is devoted for conclusion.

2. EXTREME ORDER STATISTICS PLOTS

2.1. Extreme Order Statistics

Let $X_1, \ldots, X_n$be a sample from a distribution function F, probability function $f(x)$ and quantile function $x(F)$. When the X_i's are arranged in ascending order of magnitude and then written as

$$X_{1:n} \leq \cdots \leq X_{n:n}$$

$X_{r;n}$ is the rth order statistic. Since the event $(X_{r:n} \leq x)$ occurs if and only if at least r of the X_i's are less than or equal to x, $F_{r:n}$ is expressible in terms of F as the binomial tail probability

$$F_{r:n}(x) = Pr(X_{r:n} \leq x) = \sum_{j=r}^{n} \binom{n}{j} F^j(x)[1 - F(x)]^{n-j}$$

The expected value of order statistics is

$$E(X_{r:n}) = r\binom{n}{r}\int_0^1 x(F)F^{r-1}(1-F)^{n-r}dF, r \leq n$$

This can be re-written as

$$E(X_{r:n}) = r\binom{n}{r}E[x(F)F^{r-1}(1-F)^{n-r}]$$

See; David (1981)

Let

$$M_{n:n} = \max\{X_1, \ldots, X_n\}$$

Denote the maximum of the first n random variables. Its distribution function is given by

$$F_{n:n} = [F(x)]^n, -\infty < x < \infty$$

As pointed out by Arnold et al. (2008), clearly knowledge of the distribution of $X_{n:n}$ determines $F(x)$ completely. This is true since

$$F(x) = [F_{n:n}]^{1/n}, -\infty < x < \infty$$

Moreover, Chan (1967) has shown that if $E|X| < \infty$ then $F(x)$ is uniquely determined by the sequence

$$\{E(X_{n:n}): n = 1,2,3, \dots\}$$

Let

$$M_{1:n} = \min\{X_1, \dots, X_n\}$$

Denote the minimum of the first n random variables. The distribution function is given by

$$F_{1:n} = 1 - [1 - F(x)]^n, -\infty < x < \infty$$

Clearly knowledge of the distribution of $X_{1:n}$ determines F completely. This is true since

$$F(x) = 1 - [1 - F_{1:n}]^{1/n}, -\infty < x < \infty$$

Also, Chan (1967) has shown that if $E|X| < \infty$ then F is uniquely determined by the sequence

$$\{E(X_{1:n}): n = 1,2,3, \dots\}$$

For example,

$$E(X_{1:n}) = 1/n, n = 1,2,3, \dots$$

if and only if F is unit exponential $(F(x) = 1 - \exp(-x), x > 0)$,

$$E(X_{n:n}) = 2n/(2n+1), n = 1,2,3, \dots$$

if and only if F is triangular $(F(x) = x^2, 0 < x < 1)$ and

$E(X_{1:n}) = 1/(2^n - 1), n = 1,2,3, \dots$

if and only if F is geometric$(P(X = x) = 2^{-x-1}, x = 0,1,2, \dots)$; see, for example, Huang (1989).

2.2. Min and Max Plots

For a given data of size n, $x_1, x_2, \dots, x_n$, the theoretical min curve based on the expected value of order statistics is defined as

$$E(X_{1:k}) = kE[x(F)(1-F)^{k-1}], k = 1,2, \dots, n$$

From Downton (1966) and Elamir and Seheult (2003) this can be estimated as

$$\hat{E}(X_{1:k}) = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n} \binom{n-i}{k-1} x_{i:n}, k = 1,2,\dots,n$$

The theoretical max curve based on the expected value of order statistics is defined as

$$E(X_{k:k}) = kE[x(F)F^{k-1}], k = 1,2,\dots,n$$

From Downton (1966) and Elamir and Seheult (2004) this can be estimated as

$$\hat{E}(X_{k:k}) = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n} \binom{i-1}{k-1} x_{i:n}. k = 1,2,\dots,n$$

Nonparametric extreme order statistics plot consists of two plots

$$\text{Min curve} := \left(k \text{ or } x_{i:n} \text{ versus } \hat{E}(X_{1:k})\right), k = 1,\dots,n, i = 1,2,\dots,n$$

This curve starts from the average $\bar{x} = \hat{E}(X_{1:1})$ to the minimum value $x_{1:n} = \hat{E}(X_{1:n})$.

Also the max curve is plotted as

$$\text{Max curve} := \left(k \text{ or } x_{i:n} \text{ versus } \hat{E}(X_{k:k})\right), k = 1,\dots,n, i = 1,2,\dots,n$$

This curve starts from the average $\bar{x} = \hat{E}(X_{1:1})$ to the maximum value $x_{n:n} = \hat{E}(X_{n:n})$.

Both curves should tell us the whole picture about the distribution function of a random variable X for a given data. Also each curve in its own should reflect all the information about the whole distribution for a random variable X for a given data.

Extreme order statistics plots can compare theoretical distribution with any data using

$$\text{Min plot} := \left(\left(E(X_{1:k}) \text{ versus } \hat{E}(X_{1:k})\right), k = 1,\dots,n\right)$$

and

$$\text{Max plot} := \left(\left(E(X_{k:k}) \text{ versus } \hat{E}(X_{k:k})\right), k = 1,\dots,n\right)$$

Also if two data come from the same distribution, the full nonparametric plot is

$$\text{Min line plot} := \left(\left(\hat{E}(X_{1:k}) \text{ versus } \hat{E}(X_{1:k})\right), k = 1,\dots,n\right)$$

and

$$\text{Max line plot} := \left(\left(\hat{E}(X_{1:k}) \text{ versus } \hat{E}(X_{1:k})\right), k = 1, \ldots, n\right)$$

In all these cases the Min and Max plots should show relationship close to straight line.

3. MINI AND MAX NORMAL PLOTS

Since a variety of estimation and inferential procedures in the practice depends on the assumption of normality, the graphical characterization of the normal distribution is very important and the most common graph is quantile quantile normal plot. The Min and Max normal plots will complete the picture of QQ-norm plot especially at the extreme tails of the distribution. The Min-norm plot is proposed by plotting the exact minimum order statistics of size k from standard normal distribution that can be obtained from package *EnvStats* in R software versus estimated minimum order statistics from a data as

$$\text{Min normal plot} := \left(\text{evNormOrdStatsScalar}(1, k) \text{ versus } \hat{E}(X_{1:k}), k = 1,2, \ldots, n\right)$$

Also the maximum plot is proposed as

$$\text{Max normal plot} := \left(\text{evNormOrdStatsScalar}(k, k) \text{ versus } \hat{E}(X_{k:k}), k = 1,2, \ldots, n\right)$$

The pattern of points in the Min-normal and Max-normal plots must show straight line or close to straight line.

Figure 1 shows Min-normal, Max-normal and normal Q-Q plots for simulated data from normal distribution (500,20) and $n = 200$. It is clear that the Min-normal gives more weights to lower tail of the distribution while the Max-normal gives more weights to upper tail of the distribution. Note also that the Min and Max normal plots are more stable than QQ-normal at the extreme tails of the distribution.

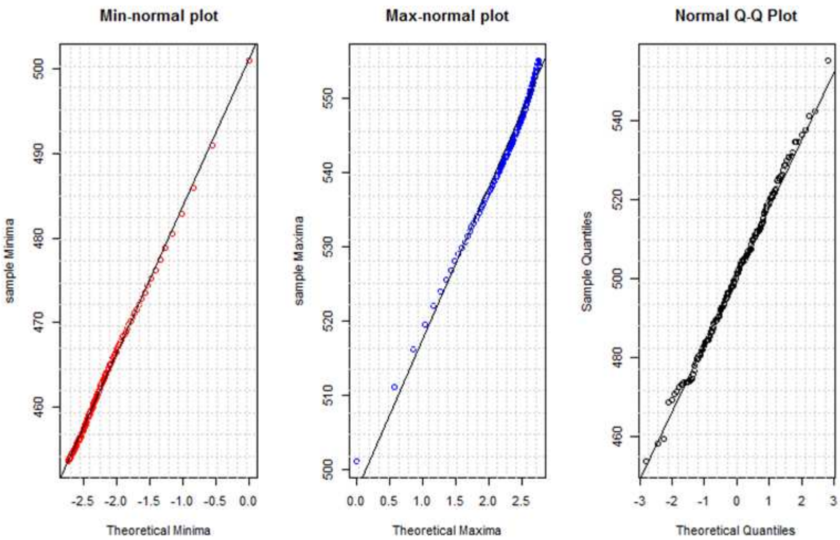

Figure 1: Min Normal, Max Normal and Q-Q Normal Plots for Simulated Data from Normal Distribution (500, 20) AND$n = 200$.

Moreover, the location and scale parameters can be estimated from Min and Max normal plots. The mean of the population can be estimated from the largest value in Min plot and the lowest value in Max plot, i.e., $\hat{\mu} \approx 501$. The Gini's measure (G) of variability can be estimated from the plot by using the highest two points in Min plot and lowest two points in Max plot where

$$G = E(Y_{2:2} - Y_{1:2}) = 2E(Y_{1:1} - Y_{1:2}) = 2E(Y_{2:2} - Y_{1:1})$$

The estimated Gini's measure is 2(501-491)=20 and 2(511-501)=20; see, Elamir (2013).

Figure 2 shows Min, Max and Q-Q normal plots for simulated data from Laplace distribution (500,20) and $n = 200$. Note that the curvature is clear in the Min plot normal. Note that R-program for Min-normal and Max-normal plots is given in Appendix A.

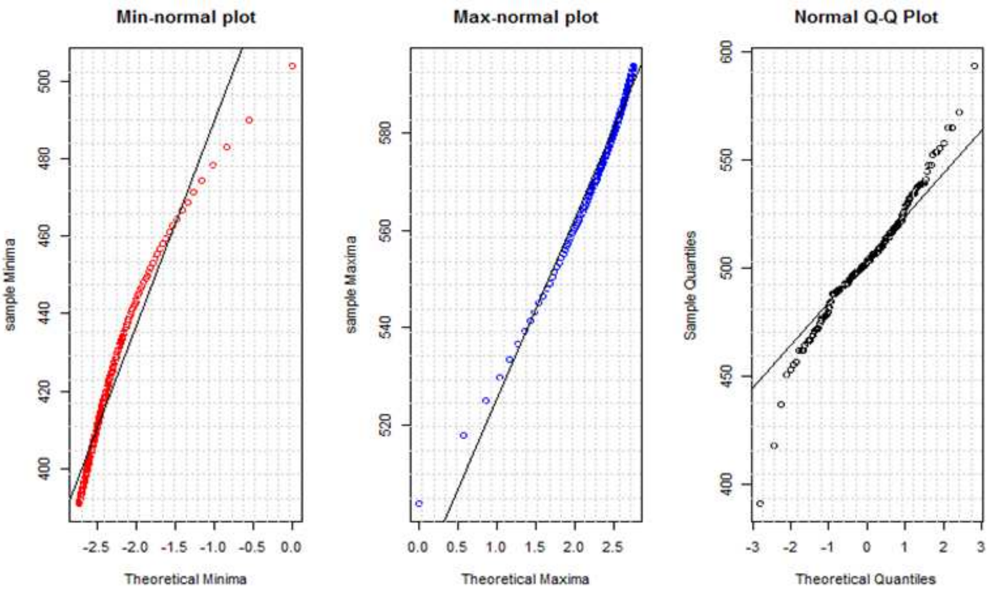

Figure 2: Min, Max and Q-Q Normal Plots for Simulated Data from Laplace Distribution (500, 20) And$n = 200$.

4. NONPARAMETRIC VISUALIZATION

The extreme order statistic plots can be used for nonparametric visualization for Gumbel and Weibull distributions as follows.

4.1. Gumbel Distribution

This distribution is used to model the distribution of the maximum or the minimum of a number of samples of various distributions. It is useful in predicting the chance that an extreme earthquake, flood or other natural disaster will occur; see, Gumbel (1954). Consider the density function for Gumbel distribution is given as

$$f(x;\alpha,\beta) = \beta^{-1}\exp[-(x-\alpha)/\beta]\exp[-\exp[-(x-\alpha)/\beta]], -\infty < x < \infty$$

and the cumulative distribution function is

$$F(x) = \exp[-\exp[-(x-\alpha)/\beta]]$$

From Arnold et al. (2008) the maximum order statistics can be obtained as

$$E(X_{k:k}) = \alpha + 0.5772\beta + \beta \log k$$

Completely nonparametric visualization for Gumbel distribution can be done as

$$\text{Max plot} := \left(\log k, \hat{E}(X_{k:k})\right), k = 1,2,\dots,n$$

Also the quantile function is

$$x(F) = \alpha - \beta \log(-\log F)$$

The quantile plot is

$$\text{quantile plot} := (\log(-\log F), x_i), i = 1,2,\dots,n$$

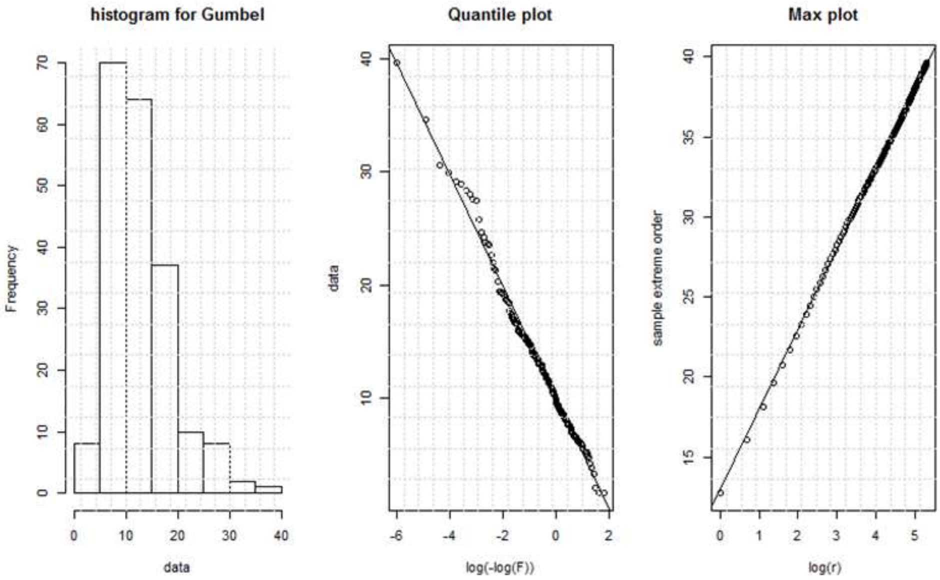

Figure 3: Histogram, Quantile and Max Plots for Simulated Data from Gumbel (10, 5) Distribution and $n = 200$

It is clear from Figure 3 the Max plot has direct straight line and the quantile plot has inverse straight line. This is a very strong indication for Gumbel distribution. Moreover, the slopes for two plots are -4.934 and 5.025 and the intercepts are 9.889 and 12.948, respectively.

4.2. Weibull Distribution

The Weibull distribution is used in many areas such as survival analysis, reliability engineering, weather forecasting and wind speed analysis; see, Johnson et al. (1994). Consider the density function for Weibull distribution is given as

$$f(x;\lambda,\delta) = \frac{\delta}{\lambda}\left(\frac{x}{\lambda}\right)^{\delta-1} e^{-(x/\lambda)^{\delta}}, x \geq 0, \delta, \lambda > 0$$

The cumulative distribution function is known to be

$$F(x) = 1 - e^{-(x/\lambda)^{\delta}}$$

The minimum and maximum order statistics can be obtained from Arnold et al. (2008) as

$$E(X_{1:k}) = \lambda\Gamma(1 + 1/\delta)k^{-1/\delta}$$

and

$$E(X_{k:k}) = \lambda\Gamma(1 + 1/\delta)\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(1+j)^{-1-1/\delta}$$

Completely nonparametric visualization for Weibull distribution may be obtained by taking the logarithm of $E(X_{1:k})$ as

$$\log E(X_{1:k}) = \log\lambda + \log\Gamma(1 + 1/\delta) - \frac{1}{\delta}\log k\,, k = 1,2,\ldots,n$$

Therefore,

$$\log \text{Min plot} := \left(\log k\,, \log\hat{E}(X_{1:k})\right)$$

This indicates that the Weibull distribution with density $f(x;\lambda,\delta) = \frac{\delta}{\lambda}\left(\frac{x}{\lambda}\right)^{\delta-1} e^{-(x/\lambda)^{\delta}}$ can be characterized by the inverse linear relationship between the logarithm of minimum order statistics and the logarithm of the ranks whatsoever the values of the parameters λ and δ. Also, this plot characterizes the exponential distribution for $\delta = 1$; i.e., the slope is 1.

The quantile function can be obtained from cumulative function as

$$\log[-\log(1-F)] = -\delta\log\lambda + \delta\log x$$

Therefore, log quantile plot is

$$\log\text{quantile} := (\log x_i, \log[-\log(1-F)]\,, i = 1,2,\ldots,n)$$

This is also known as Weibull plot; see, Johnson et al. (1994). This indicates that the Weibull distribution with density $f(x;\lambda,\delta) = \frac{\delta}{\lambda}\left(\frac{x}{\lambda}\right)^{\delta-1} e^{-(x/\lambda)^{\delta}}$ can be characterized by the direct linear relationship between $\log[-\log(1-F)]$ and $\log x_i$.

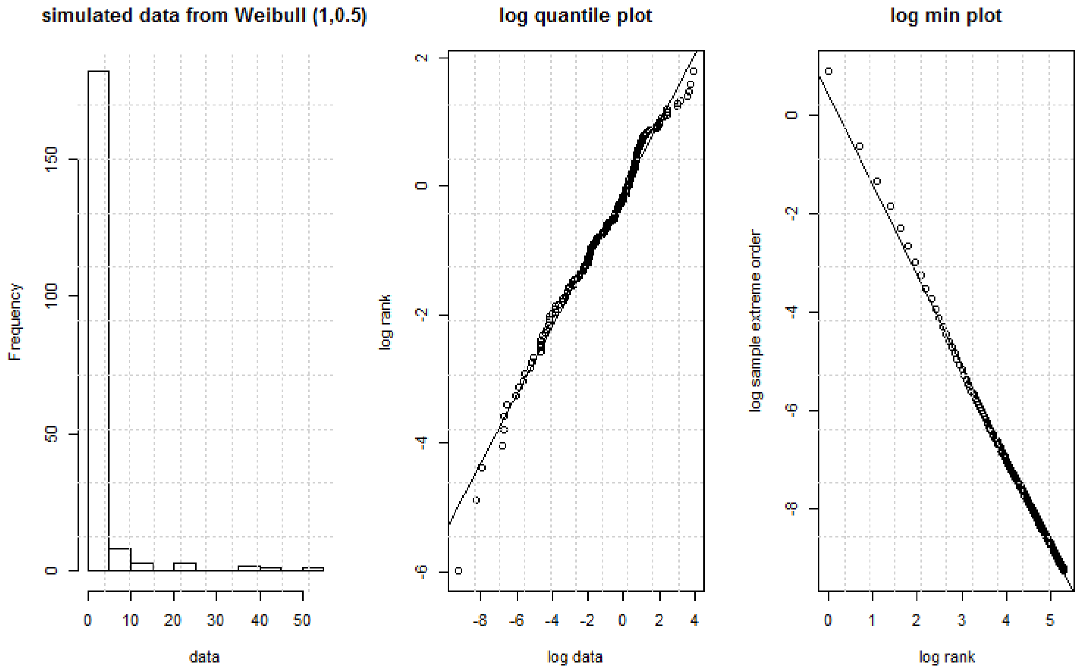

Figure 4: Histogram, Log Quantile and Log Min Plots for Simulated Data from Weibull (1, 0.5) Distribution and $n = 200$

It is clear from Figure 4 the log min plot has inverse straight line and log quantile plot has a direct straight. This is a very strong indication for Weibull distribution. Moreover, the slopes for two plots are -1.85 and 0.53 and the intercepts are 0.445 and -0.062, respectively.

5. DISCRETE DATA

The Min-Max plots have advantage to be applied for discrete distributions to graphically and informally test the goodness-of-fit of a sample in an exploratory way.

The binomial distribution with parameters m and p is the discrete probability distribution of the number of successes in a sequence of m independent yes/no trials each of which yields success with probability p. The probability mass function is

$$f(x;m,p) = \binom{m}{x} p^x (1-p)^{m-x}, x = 0,1,\dots,m$$

and cumulative

$$F(x) = \sum_{j=0}^{x} \binom{m}{j} p^j (1-p)^{m-j}$$

From Arnold et al. (2008) the minimum and maximum order statistics can be obtained as

$$E(X_{1:k}) = \sum_{x=0}^{m-1} [1 - F(x)]^k$$

and

$$E(X_{k:k}) = \sum_{x=0}^{m-1} \{1 - [F(x)]^k\}$$

Bernoulli distribution is a special case of binomial distribution at $m = 1$ where a random variable which takes the value 1 with success probability of p and the value 0 with failure probability of $q = 1 - p$. The minimum and maximum order statistics can be obtained in a simple form for Bernoulli distribution as

$$E(X_{1:k}) = p^k \text{ and } E(Y_{k:k}) = 1 - q^k, k = 1,2,\dots,n$$

For given p, the proposed plot for Bernoulli distribution is

$$\text{Min plot} := \left(p^k \text{ versus } \hat{E}(X_{1:k}), k = 1,2,\dots,n\right)$$

and

$$\text{Max plot} := \left(1 - q^k \text{ versus } \hat{E}(X_{k:k}), k = 1,2,\dots,n\right)$$

Figure 5 shows Min and Max plots for simulated data from Bernoulli distribution (p=0.5) and $n = 100$ versus theoretical Min and Max values p^k and $1 - q^k$. Also, Figure 6 shows Min and Max plots for simulated data from Bernoulli distribution (p=0.05). Both graphs show straight lines.

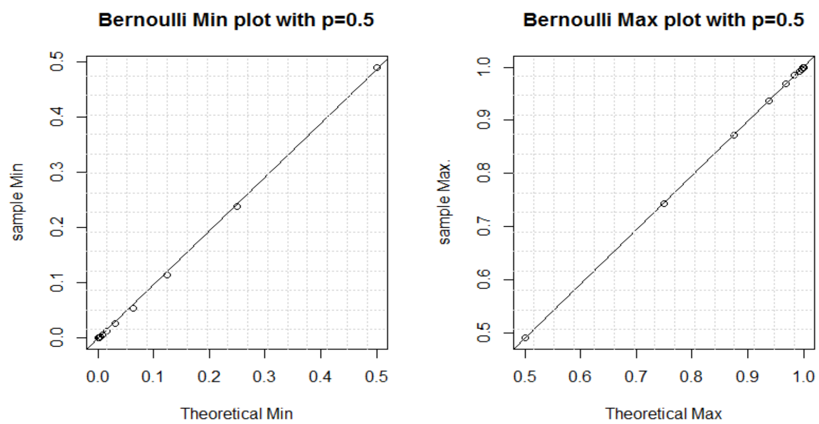

Figure 5: Min and Max Plots for Simulated Data from Bernoulli Distribution (P=0.5) versus the Oretical $0.5^k$and $1-(0.5)^k$And$n = 100$

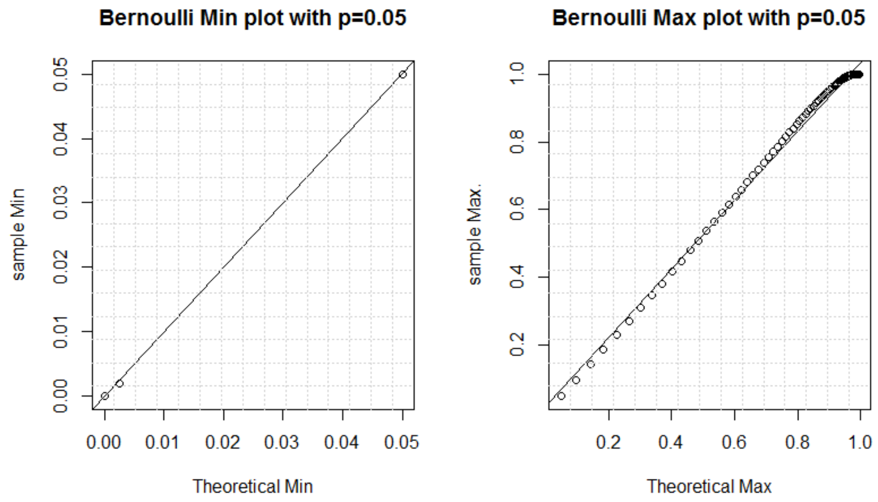

Figure 6: Min and Max Plots for Simulated Data from Bernoulli Distribution (P=0.05) versus the Theoretical Versus Theoretical $\mathbf{0.05^k}$and $\mathbf{1-(0.95)^k}$And$\mathbf{n = 100}$

Figure 7 shows Min and Max plots for simulated data from Bernoulli distribution (p=0.5) versus the theoretical $0.80^k$and $1-(0.20)^k$and $n = 100$. It is clear that the data does not come from Bernoulli distribution.

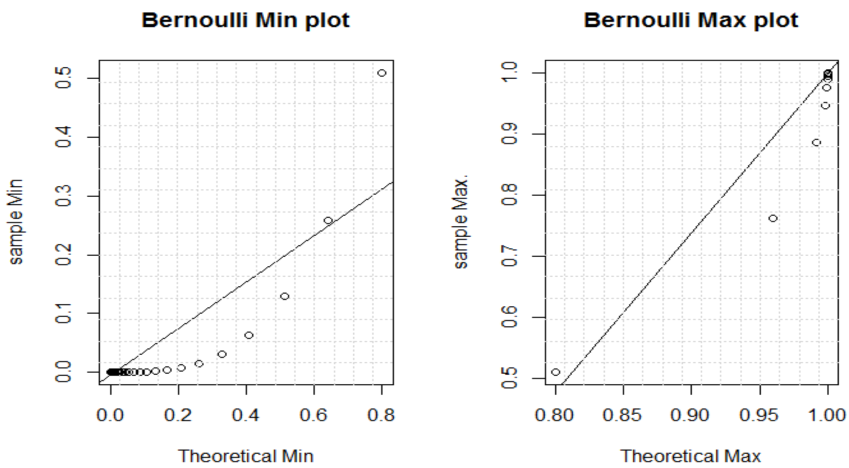

Figure 7: Min and Max Plots for Simulated Data from Bernoulli Distribution (P=0.5) Versus the Theoretical Versus Theoretical $\mathbf{0.80^k}$and $\mathbf{1-(0.20)^k}$and $\mathbf{n = 100}$

The geometric distribution that is used for modeling the number of trials up to and including the first success that requires x number of independent trials each with success probability p is defined as

$$f(x) = pq^{x-1}, x = 1,2, \ldots.$$

From Margolin and Winokur (1967) the Min order statistics can be obtained as

$$E(X_{1:k}) = \frac{1}{1-q^k}$$

and Max order statistics

$$E(X_{k:k}) = k\sum_{j=0}^{k-1}\frac{(-1)^j\binom{k-1}{j}}{(j+1)(1-q^{j+1})}$$

Therefore, for given p the proposed Min and Max plots for geometric distribution are

$$\text{Min plot} := \left(\frac{1}{1-q^k}, \hat{E}(X_{1:k}), k = 1,2,\dots,n\right)$$

and

$$\text{Max plot} := \left(k\sum_{j=0}^{k-1}\frac{(-1)^j\binom{k-1}{j}}{(j+1)(1-q^{j+1})}, \hat{E}(X_{k:k}), k = 1,2,\dots,n\right)$$

6. APPLICATION

6.1. Application 1

An experiment was performed to determine whether two forms of iron (Fe^{2+} and Fe^{3+}) are retained differently. If one form of iron were retained especially well, it would be the better dietary supplement. The investigators divided 36 mice randomly into two groups of 18 each. The mice were given iron at concentration 1.2 millimolar for both groups and later time count was taken for each mouse, and the percentage of iron retained was calculated; see, Rice (1995). The data are given in Table 1. Are these data come from the same distribution?

Table 1: The Percentage of Iron Retained at Concentration 1.2 Milli molar

Y=Fe^{3+}	2.2	2.93	3.08	3.49	4.11	4.95	5.16	5.54	5.68
	6.25	7.25	7.90	8.85	11.96	15.54	15.89	18.3	18.59
Y1= Fe^{2+}	4.04	4.16	4.42	4.93	5.49	5.77	5.86	6.28	6.97
	7.06	7.78	9.23	9.34	9.91	13.46	18.4	23.89	26.39

Figure 8 shows the Min line, Max line and QQ plots for these data. The mean can be obtained from the graph as 8.20 and 9.60, respectively. Also the gini's measures are $2(11.2-8.2)=6$ and $2(13-9.6)=6.8$.,respectively. The plots indicate that the data are right skewed and do not come from the same distribution.

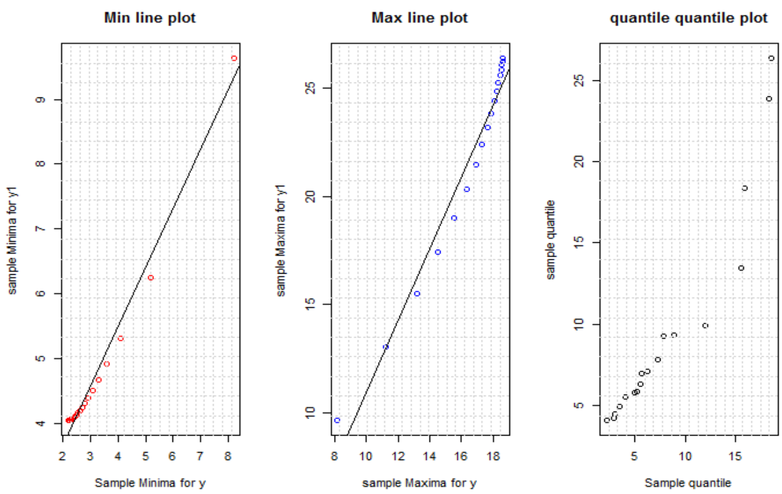

Figure 8: Min Line, Max Line and QQ Plots for the Percentage of Iron Retained at Concentration 1.2 Milli Molar

6.2. Application 2: Pareto Distribution

Pareto distribution represents one of the most famous distributions and it is widely used in economics, finance and natural sciences; see Johnson et al. (1994) and Haseeb et al. (2012). The density for Pareto I is defined as

$$f(x;\delta) = \alpha\beta^{\alpha}x^{-\alpha-1}, x \geq \beta \geq 0, \alpha > 0$$

Where β is the scale and α is the shape parameter and the smaller α, the fatter the right tail of the distribution. For $\alpha \leq 2$ the Pareto distribution has infinite variance. For $\alpha \leq 1$ the expected value does not exist. Figure 9 shows Min line, Max line and QQ plots for simulated data from ParetoI (10,3) for two variables y and y_1 and $n = 180$. It is clear that the stability of Min and Max plots over QQ plot especially at the extreme tails.

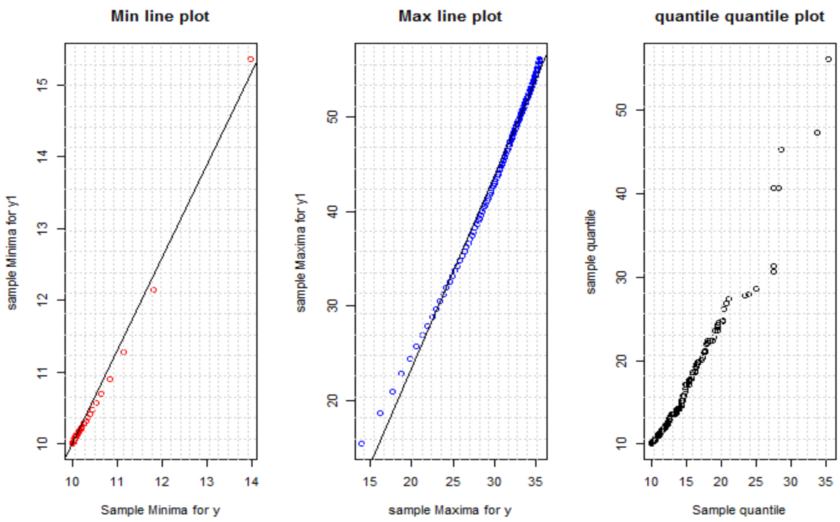

Figure 9: Min Line, Max Line and QQ Plots for Simulated Data from Paretoi (10, 3) for Both y and y_1 and $n = 180$

7. CONCLUSIONS

Min and Max plots based on minimum and maximum order statistic of size k are proposed in nonparametric and parametric ways. These plots are very useful especially for heavy tailed distributions where they give more weights for the extreme tails. It has been shown that the Min and Max plots characterize the Gumbel and Weibull distribution non-parametrically using simple linear regression.

Since the normal distribution is very important in practice, the Min-normal and Max-normal plots are introduced and it has been shown that they had completed the picture of the data with QQ plot especially at the extreme tails of the distribution. One more advantage of Min and Max plots is that they had extended to discrete distributions such as Bernoulli, Binomial and geometric to graphically and informally test the goodness-of-fit of a sample in an exploratory way.

One limitation of Min and Max plots is when the extreme order statistics are not defined. But the Min and Max plots may still be plotted using the available information and ignoring undefined values. Of course, in this case some information will be lost.

Appendix A: R-program for Min and Max normal plots

```
library(EnvStats)
library(VGAM)
par(mfrow=c(1,3))      ### 3 graphs in one page
LGd=function(x,t){       ### function for estimated Min order statistics
 n=length(x); i=1:n; x=sort(x)
 c1=1/choose(n,t)
 t1=choose(n-i,t-1)*x
 c1*sum(t1)}

LGo=function(x,t){      ### function for estimated Max order statistic
 n=length(x); i=1:n; x=sort(x)
 c1=1/choose(n,t)
 t1=choose(i-1,t-1)*x
 c1*sum(t1)}

n=200; k=1:n;  y=rnorm(n,500,20) ### simulated normal data
wdy=0; woy=0; E11=0; Ekk=0
for (i in 1:n){
wdy[i]=LGd(y,i); woy[i]=LGo(y,i) ### estimated Min and Max order stat.
E11[i]=evNormOrdStatsScalar(1,i)  ### exact Min order statist.
Ekk[i]=evNormOrdStatsScalar(i,i) }  ### exact Max order stat
plot(E11,wdy,main="Min-normal plot",col="red",
  xlab="Theoretical Minima", ylab="sample Minima") ### Min normal plot
M1=lm(wdy~E11); abline(M1)                  ### fitting straight line
plot(Ekk,woy,main="Max-normal plot",col="blue",
   xlab="Theoretical Maxima", ylab="sample Maxima") ### Max normal plot
 M2=lm(woy~Ekk); abline(M2)               ### fitting straight line
qqnorm(y);  qqline(y)                     ### Q-Q normal plot
```

REFERENCES

1. Arnold, B. C, Balakrishnan, N. and Nagataja, H.N. (2008). *A First course in Order Statistics*. 2nd Ed., Society for Industrial and Applied Mathematics, SIAM.
2. Chambers, J., Cleveland, W., Kleiner, B. and Tukey, P. (1983). *Graphical Methods for Data* Analysis, 1st Ed., Wadsworth.
3. Chan, L.K. (1967). On a characterization of distributions by expected values of extreme order statistics. *American Mathematical Monthly*, 74, 950-951.
4. Cleveland, W.S. (1994). *The Elements of Graphing Data*. 1st Ed., Hobart Press
5. Cleveland, W. and McGill, M. (1988). *Dynamic Graphics for Statistics.* Wadsworth
6. David, H. A. (1981). *Order Statistics.* 2nd ed., Wiley, New York.
7. Downton, F. (1966). Linear estimates with polynomial coefficients. *Biometrika*, 53,129-141.
8. Elamir, E.A.H. and Seheult, A. (2003). Trimmed L-moments. *Computational Statistics and Data Analysis*, 43, 299-314.

9. Elamir, E.A.H. and Seheult, A. (2004). Exact variances of sample L-moments. *Journal of Statistical Planning and Inference*, 124, 337-359.

10. Elamir, E A. H. (2013). On estimation of some abbreviated social welfare measure. *Quality & Quantity: International Journal of Methodology*, 63, 245-268.

11. Gumbel, E.J. (1954). Statistical Theory of Extreme Values and Some Practical Applications. Applied Mathematics Series **33** (1st ed.). U.S. Department of Commerce, National Bureau of Standards.

12. Haseeb A., Khwaja, S.K. and Nayabuddin (2012). Expectation identities of Pareto distribution based on generalized order statistics and its characterization. *American Journal of Applied Mathematical Sciences*, 1, 23-29.

13. Haung, J.S. (1989). Moment problem of order statistics. A review. *International Statistical Review*, 57, pp. 59-66.

14. Johnson, Norman L.; Kotz, Samuel; Balakrishnan, N. (1994). *Continuous Univariate Distributions.* Vol. 1, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics (2nd ed.), New York: John Wiley & Sons.

15. Makkonen, L. (2008). Bringing closure to the plotting position controversy. *Communications in Statistics - Theory and Methods*, 37, 460–467

16. Margolin, B. H., H. S. Winokur Jr. (1967). Exact moments of the order statistics of the geometric distribution and their relation to inverse sampling and reliability of redundant systems. *Journal of American Statistical Association*, 62 915–925.

17. Rice, J.A. (1995). *Mathematical Statistics and Data Analysis*. 1st ed., International Thomson publishing, Duxbury press.

18. Scott, D. (1992). *Multivariate Density Estimation: Theory, Practice, and Visualization.* John Wiley and Sons.

19. Wilk, M.B.; Gnanadesikan, R. (1968). Probability plotting methods for the analysis of data. *Biometrika*, **55**, 1–17.